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Quantum deformation of KdV hierarchies and their exact solutions: q-deformed solitons

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Abstract. A complete procedure is given by which one can construct q-deformed KdV hierarchies and obtain their exact solutions with arbitrarily many q-deformed solitons. As an elementary example, we obtain the exact solution of the q-deformed single soliton of the simplest (3,2) system, i.e. the q-deformed KdV equations, and investigate this solution in the case of small deformation. It is interesting that this method can supply many exact solutions for systems of new conventional nonlinear partial differential equations.

The theory of solitons [1] is an attractive and exciting field. It brings together many branches of mathematics, some of which have touched on deep ideas. Recently, a lot of research work demonstrated that the Korteweg–de Vries (KdV) hierarchies and their soliton theory are related to the popular topics in theoretical physics: the non-perturbative two-dimensional quantum gravity, the matrix models, the two-dimensional topological field theories, the conformal field theory and the W algebra. Moreover, the theory of soliton has been widely applied to physical sciences, such as physics of fluids, optical fibers, biophysics, superconductor physics, etc. The soliton theory deals with the rare cases in which nonlinear partial differential equations are exactly solvable. We are only familiar, and for a long time, with exact soliton solutions of differential equations with conventional differential operators. We know, however, that there exists a kind of extended differential operator, the q-deformed differential operator [2]. Does there exist a 'q-deformed differential equation', which has soliton-like exact solutions? In this paper we will study this problem and give the q-deformed KdV equation and its q-deformed soliton exact solutions.

At first we review the q-deformed KdV equation proposed by Zhang [3]. We introduce a dilation operator Q

$$Qf(x) = f(xq) \tag{1}$$

and a q-deformed differential operator [2]

$$\tilde{D} = \frac{1}{(1-q^{-2})x}(1-Q^{-2}) \tag{2}$$

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which comes back to the conventional differential operator ∂ when q approaches 1, where q is a real (or complex) number but $q^2 \neq 1$. According to the definition of the $\tilde{D}$, one can prove that the q-deformed Leibniz rule is

$$\tilde{D}(f(x)g(x)) = (\tilde{D}f(x))g(x) + (Q^{-2}f(x))(\tilde{D}g(x)) \tag{3a}$$

which can be expressed in an operator form,

$$\tilde{D} \circ f = f^{(0,-2)}\tilde{D} + f^{(1,0)} \tag{3b}$$

where $\circ$ indicates that the $\tilde{D}$ before $\circ$ acts on the other functions behind $f(x)$. Hereafter we introduce a symbol

$$f^{(n,m)}(x) = (\tilde{D}^n Q^m f(x)). \tag{4}$$

One can generalize the q-deformed Leibniz rule (3b) to higher-order cases,

$$\tilde{D}^n \circ f(x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} q^{2m(n-m)} f^{(m,2m-2n)}(x)\tilde{D}^{n-m} \tag{5}$$

and to cases of negative order,

$$\tilde{D}^{-n} \circ f(x) = \sum_{m=0}^{\infty}(-1)^m \begin{bmatrix} n+m-1 \\ m \end{bmatrix} q^{-m(m+1)} f^{(m,2m+2n)}(x)\tilde{D}^{-n-m} \tag{6}$$

where

$$\begin{bmatrix} n \\ m \end{bmatrix} = \frac{[n]!}{[m]![n-m]!} \qquad [m]! = [m][m-1]\cdots[2][1] \qquad [m] = \frac{1-q^{-2m}}{1-q^{-2}} \tag{7}$$

and

$$[0]! = [0] = 0! = 1. \tag{8}$$

The q-deformed KdV hierarchy is described in terms of the q-deformed pseudo-differential operator which is a formal expression

$$K = \sum_{n=-\infty}^{M} k_n \tilde{D}^n \tag{9}$$

where the coefficients $k_n(x)$ are functions of a variable x and $\tilde{D}$ is defined in equation (2). The multiplicative rule of two q-deformed pseudo differential operators has been given by equations (5) and (6). Therefore the q-deformed pseudo-differential operators form a closed algebra. One further introduces the decomposition

$$K = K_+ + K_- \qquad K_+ = \sum_{n=0}^{M} k_n \tilde{D}^n \qquad K_- = \sum_{n=1}^{\infty} k_{-n}\tilde{D}^{-n}. \tag{10}$$

The Nth q-deformed KdV hierarchy [4] consists of an infinite set of q-deformed differential equations with commuting flows, where the equations are about the coefficients $V_n(x, t_p)$ ($n = 0, 1, \cdots, N-1$) of a q-deformed differential operator L of order N that has been put in the canonical form

$$L = \tilde{D}^N + \sum_{n=0}^{N-1} V_n \tilde{D}^n. \tag{11}$$

In the algebra of q-deformed pseudo differential operators, L has an unique Nth root $L^{1/N}$, and in the Lax representation [5] the pth flow of the Nth q-deformed KdV hierarchy, known as the (p, N) system, is given by

$$\frac{\partial}{\partial t_p} L = [(L^{p/N})_+, L] = [L, (L^{p/N})_-] \tag{12}$$

where $\{t_p\}$ are called time parameters.

The simplest system of (p, N) q-deformed KdV hierarchies must be the (3,2) system, known as q-deformed KdV equations. Let us choose this system in order to illustrate the above procedure. This model is obtained by taking L to be a q-deformed differential operator of order two:

$$L = \tilde{D}^2 + V_1(x,t)\tilde{D} + V_0(x,t). \tag{13}$$

The formal expansion of $L^{1/2}$ in powers of $\tilde{D}$ is given by

$$L^{1/2} = \tilde{D} + \sum_{n=0}^{\infty} W_{-n}\tilde{D}^{-n}. \tag{14}$$

Since one later needs only the first five of the coefficients W_{-n} in the q-deformed KdV equations, one gives them in terms of V_1 and V_0, order by order:

$$W_0 = (1+Q^{-2})^{-1}V_1 = \sum_{n=0}^{\infty}(-1)^n V_1^{(0,-2n)} \tag{15}$$

$$W_{-1} = -(1+Q^{-2})^{-1}(-V_0 + W_0^{(1,0)} + W_0^2) \tag{16}$$

$$W_{-2} = -(1+Q^{-2})^{-1}(W_{-1}W_0^{(0,2)} + W_{-1}^{(1,0)} + W_0W_{-1}) \tag{17}$$

$$\begin{aligned} W_{-3} = -(1+Q^{-2})^{-1}(&-q^{-2}W_{-1}W_0^{(1,4)} + W_{-2}W_0^{(0,4)} \\ &+W_{-1}W_{-1}^{(0,2)} + W_{-2}^{(1,0)} + W_0W_{-2}) \end{aligned} \tag{18}$$

$$\begin{aligned} W_{-4} = -(1+Q^{-2})^{-1}(&q^{-6}W_{-1}W_0^{(2,6)} - q^{-2}[2]W_{-2}W_0^{(1,6)} + W_{-3}W_0^{(0,6)} \\ &-q^{-2}W_{-1}W_{-1}^{(1,4)} + W_{-2}W_{-1}^{(0,4)} + W_{-1}W_{-2}^{(0,2)} + W_{-3}^{(1,0)} + W_0W_{-3}). \end{aligned} \tag{19}$$

The q-deformed differential operator that appeared in the (3,2) system is $(L^{3/2})_+$. Due to the last identity of equation (12), one needs only the first two terms of the q-deformed pseudo-differential operator $(L^{3/2})_-$,

$$(L^{3/2})_- = U_{-1}\tilde{D}^{-1} + U_{-2}\tilde{D}^{-2} + \cdots \tag{20}$$

where the coefficients U_{-1} and U_{-2} are given by the following expressions:

$$U_{-1} = W_{-3} - q^{-2}W_{-1}V_1^{(1,4)} + W_{-2}V_1^{(0,4)} + W_{-1}V_0^{(0,2)} \tag{21}$$

$$\begin{aligned} U_{-2} = W_{-4} &+ q^{-6}W_{-1}V_1^{(2,6)} - q^{-2}[2]W_{-2}V_1^{(1,6)} + W_{-3}V_1^{(0,6)} \\ &-q^{-2}W_{-1}V_0^{(1,4)} + W_{-2}V_0^{(0,4)}. \end{aligned} \tag{22}$$

Now we can obtain the q-deformed KdV equations by using equation (12) (let $t_3 = t$) and equation (20),

$$\frac{\partial V_1}{\partial t} = U_{-1}^{(0,-4)} - U_{-1} \tag{23}$$

$$\frac{\partial V_0}{\partial t} = (U_{-2}^{(0,-4)} - U_{-2}) + V_1(U_{-1}^{(0,-2)} - U_{-1}) - U_{-1}(V_1^{(0,-2)} - V_1) + q^2[2]U_{-1}^{(1,-2)}. \tag{24}$$

By using equations (21 and (22) and (15)–(19), the right sides of equations (23) and (24) can ultimately be expressed in terms of pure V_1 and V_0. When q approaches 1, we can expand the q-deformed KdV equation and its solution in the first order in $\epsilon = 1 - q$ (see section 6 of [3]),

$$V_0 = Y + \epsilon(Y - Z) \tag{25}$$

$$V_1 = 2\epsilon x Y \tag{26}$$

$$\frac{\partial Y}{\partial t} = \tfrac{1}{4}(Y''' + 6YY') \tag{27}$$

$$\frac{\partial Z}{\partial t} = \tfrac{1}{4}(Z''' + 6YZ' + 6Y'Z) + \tfrac{3}{2}(Y''' + 2YY') \tag{28}$$

where $Z' = \partial Z/\partial x$ becomes a conventional derivative. It is clear that the q-deformed KdV equation returns to the ordinary KdV equation in the limit $q \to 1$.

Now we can construct a q-differential operator L of order N whose coefficients are functions of variables x and t_p and satisfy equation (12) [6]. Let M be an arbitrary positive integer, i.e. the number of solitons. Let

$$\begin{aligned} & y_k(x, t_p) = \exp_q(\alpha_k x) \cdot \exp({\alpha_k}^p\, t_p) + a_k \exp_q(\rho\alpha_k x) \cdot \exp(\rho^p {\alpha_k}^p t_p) \\ & k = 1, \cdots, M \end{aligned} \tag{29}$$

where $\{\alpha_k\}$, $\{a_k\}$ and $(k = 1, \cdots, M)$ are complex constants, $\alpha_k \neq \alpha_l$ for $k \neq l$, $\rho^N = 1$, and

$$\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \tag{30}$$

which is the so-called q-deformed exponential function. Let

$$\phi = \frac{1}{\Delta} \begin{vmatrix} y_1 & \cdots & y_M & 1 \\ y_1^{(1,0)} & \cdots & y_M^{(1,0)} & \tilde{D} \\ \vdots & & \vdots & \vdots \\ y_1^{(M-1,0)} & \cdots & y_M^{(M-1,0)} & \tilde{D}^{M-1} \\ y_1^{(M,0)} & \cdots & y_M^{(M,0)} & \tilde{D}^M \end{vmatrix} \tag{31}$$

where Δ is the q-deformed Wronskian of $y_1, \cdots, y_M$, i.e. the minor of $\tilde{D}^M$. In the expansion of the determinant by the elements of the last column, $\tilde{D}^i$ must be written to the right of the minors. Then ϕ is a q-deformed differential operator of order M with the highest coefficient 1. It is obvious that the functions y_k have the following properties:

$$\frac{\partial}{\partial t_p} y_k = \tilde{D}^p y_k \qquad \tilde{D}^N y_k = \alpha_k^N y_k \qquad \phi y_k = 0. \tag{32}$$

Now we construct operator L by 'dressing' the operator $\tilde{D}^N$ using the operator ϕ:

$$L = \phi \tilde{D}^N \phi^{-1}. \tag{33}$$

Following Dickey step by step (see page 16 of [6]), one can prove that $L_- = 0$ and that L indeed satisfies equation (12). Therefore, the coefficients of L give the exact solutions of the (p, N) type q-deformed KdV equation.

As an elementary example, let us look for a single-soliton solution of the q-KdV equation. We take $M = 1$, $N = 2$, $p = 3$, $\rho = -1$, which gives

$$y_1 = \exp_q(\alpha x) \cdot \exp(\alpha^3 t) + a \exp_q(-\alpha x) \cdot \exp(-\alpha^3 t). \tag{34}$$

Shifting the origin of t, one can set $a = 1$ without loss of generality:

$$y_{(+)} = \exp_q(\alpha x) \cdot \exp(\alpha^3 t) + \exp_q(-\alpha x) \cdot \exp(-\alpha^3 t). \tag{35}$$

According to equation (31),

$$\phi = \tilde{D} - y_{(+)}^{(1,0)}/y_{(+)}. \tag{36}$$

If we introduce the expressions

$$y_{(-)} = \exp_q(\alpha x)\cdot\exp(\alpha^3 t) - \exp_q(-\alpha x)\cdot\exp(-\alpha^3 t) \tag{37}$$

and

$$u_0 = y_{(-)}/y_{(+)} \tag{38}$$

then

$$\phi = \tilde{D} - \alpha u_0. \tag{39}$$

The inverse of ϕ can be calculated as

$$\begin{aligned}\phi^{-1} = \tilde{D}^{-1} &+ \alpha u_0^{(0,2)}\tilde{D}^{-2} + \alpha^2(-q^{-2}\alpha^{-1}u_0^{(1,4)} + u_0^{(0,2)}u_0^{(0,4)})\tilde{D}^{-3}\\ &+\alpha^3(q^{-6}\alpha^{-2}u_0^{(2,6)} - q^{-2}\alpha^{-1}u_0^{(1,4)}u_0^{(0,6)} - q^{-2}[2]\alpha^{-1}u_0^{(0,2)}u_0^{(1,6)}\\ &+u_0^{(0,2)}u_0^{(0,4)}u_0^{(0,6)})\tilde{D}^{-4} + \cdots .\end{aligned} \tag{40}$$

Thus we can obtain the operator L:

$$\begin{aligned}L = \phi\tilde{D}^2\phi^{-1} = \tilde{D}^2 &+ \alpha(u_0^{(0,-4)} - u_0)\tilde{D} + \alpha^2\{(1+q^2)\alpha^{-1}u_0^{(1,-2)} + u_0^{(0,-2)}(u_0^{(0,-4)} - u_0)\}\\ &+\alpha^3\{\alpha^{-2}u_0^{(2,0)} + \alpha^{-1}u_0^{(1,0)}(u_0^{(0,-4)} - u_0) + (1+q^2)\alpha^{-1}u_0u_0^{(1,-2)}\\ &+u_0u_0^{(0,-2)}(u_0^{(0,-4)} - u_0)\}\tilde{D}^{-1} + \cdots .\end{aligned} \tag{41}$$

According to (13), the exact solution of the q-deformed KdV equations (23), (24) is

$$V_1(x,t) = \alpha(u_0^{(0,-4)} - u_0) \tag{42}$$

$$V_0(x,t) = \alpha^2\{(1+q^{-2}) - u_0u_0^{(0,-2)} - q^{-2}u_0^{(0,-2)}u_0^{(0,-4)}\}. \tag{43}$$

Although of the exact analytic expression of the q-deformed soliton has been obtained, the dependence of soliton behaviour on the parameter q is rather complicated, and we shall investigate this problem in another paper. Owing to $L_- = 0$, we find infinitely many identities, the first of which, stating that the coefficient of $\tilde{D}^{-1}$ must vanish, is

$$\begin{aligned}&q^{-2}u_0 - (1+q^{-2})u_0^{(0,-2)} + u_0^{(0,-4)}\\ &+u_0^{(0,-2)}(u_0u_0^{(0,-2)} - (1+q^{-2})u_0u_0^{(0,-4)} + q^{-2}u_0^{(0,-2)}u_0^{(0,-4)}) = 0.\end{aligned} \tag{44}$$

This identity comes from $(L_+\phi - \phi\tilde{D}^2)y_{(+)} = 0$. We have checked this identity by using numerical calculation of random sampling. A direct analytical proof requires a development of some new calculating techniques for q-deformed exponential functions.

In order to compare the q-deformed solution with the ordinary one, we consider the case of small $\epsilon = 1 - q$. We have, up to first order in ϵ,

$$[n]! = n!\,(1 + \tfrac{1}{2}n(n+1)\epsilon) \tag{45}$$

$$\exp_q(x) = \mathrm{e}^x(1 - \tfrac{1}{2}\epsilon x^2) \tag{46}$$

$$V_1(x,t) = \frac{4\epsilon\alpha^2 x}{\cosh^2(\alpha x + \alpha^3 t)} \tag{47}$$

$$V_0(x,t) = \frac{2\alpha^2(1+\epsilon)}{\cosh^2(\alpha x + \alpha^3 t)} - 8\epsilon\alpha^3 x\frac{\sinh(\alpha x + \alpha^3 t)}{\cosh^3(\alpha x + \alpha^3 t)}. \tag{48}$$

When $\epsilon = 0$, $V_1(x,t)$ vanishes and $V_0(x,t)$ returns to the standard soliton solution. Comparing equations (47), (48) with equations (25), (26) we obtain

$$Y(x,t) = \frac{2\alpha^2}{\cosh^2(\alpha x + \alpha^3 t)} \tag{49}$$

$$Z(x,t) = 8\alpha^3 x \frac{\sinh(\alpha x + \alpha^3 t)}{\cosh^3(\alpha x + \alpha^3 t)} \tag{50}$$

which give precisely the exact solution of equations (27) and (28).

It is well known that solving a system of nonlinear partial differential equations exactly is a very difficult task. However, the q-deformed KdV hierarchies and their exact solutions provide an opportunity. When we expand the q-deformed KdV equation in powers of ϵ, we can obtain a series of the systems of nonlinear partial differential equations. Moreover, when we expand the exact solutions of the q-deformed equation as well, we obtain the exact solutions of these systems of conventional nonlinear partial differential equations. We have calculated the exact two soliton-like solutions of the q-deformed KdV equation and obtained the exact general solution of equation (28) from the mentioned two q-deformed soliton solutions with four free parameters by using the procedure suggested in this paper, but its expression is too long to be published (needs 10 pages). We only give a special example,

$$Y(x,t) = \frac{24(e^{2t+2x} + 4e^{16t+4x} + 6e^{18t+6x} + 4e^{20t+8x} + e^{34t+10x})}{\left(1 + 3e^{2t+2x} + 3e^{16t+4x} + e^{18t+6x}\right)^2} \tag{51}$$

$$\begin{aligned} Z(x,t) = 96x(&-e^{2t+2x} + 3e^{4t+4x} - 8e^{16t+4x} - 9e^{18t+6x} - 29e^{20t+8x} + 24e^{32t+8x} \\ &-24e^{22t+10x} + 29e^{34t+10x} + 9e^{36t+12x} + 8e^{38t+14x} - 3e^{50t+14x} + e^{52t+16x}) \\ &\times \left(1 + 3e^{2t+2x} + 3e^{16t+4x} + e^{18t+6x}\right)^{-3} \end{aligned} \tag{52}$$

in order for the reader to check easily, by using MATHEMATICA, that they are indeed the exact solution of the system of equations (27), (28). Here we see that the q-deformed parameter q has disappeared. Therefore, it is interesting that, using the method of the q-deformed KdV hierarchy presented here, one can solve exactly many systems of new conventional nonlinear partial differential equations.

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